8-4 Magnetostatic Boundary Value Problems

Reading Assignment: *pp. 260-263*

Q:

A:

We must solve differential equations, **and** apply boundary conditions to find a **unique** solution.

Good news! In electrical and computer engineering, the sources $\mathbf{J}(\bar{r})$ are typically **known** (unlike sources $\rho_v(\bar{r})$).

This process is best demonstrated with an **example**:

Example: Magnetostatic Boundary Conditions

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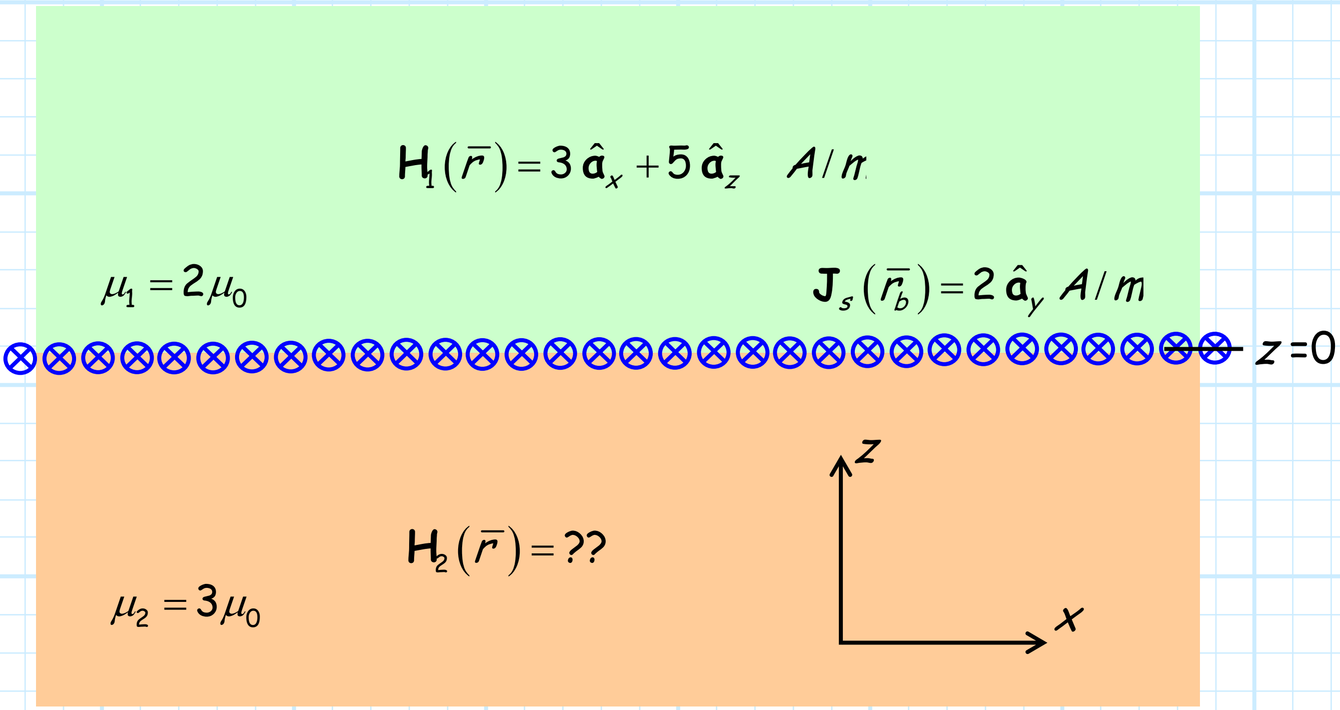
Consider two **magnetic** materials, separated by some **boundary**:



Throughout region 1, there is a **constant** magnetic field:

$$\mathbf{H}_1(\bar{r}) = 3\,\hat{\mathbf{a}}_x + 5\,\hat{\mathbf{a}}_z \qquad (z > 0)$$

On the **interface** (i.e., boundary) between the two regions, there flows a **surface current**:

$$\mathbf{J}_s(\bar{r}) = \begin{cases} 0 & z > 0 \\ 2\,\hat{\mathbf{a}}_y & z = 0 \\ 0 & z < 0 \end{cases} \qquad \left[A/m\right]$$

Q: *What is* $\mathbf{H}_2(\bar{r})$ *and* $\mathbf{B}_2(\bar{r})$ *in region 2 ??*

A: Let's apply the **boundary conditions** and find out!

At the interface (i.e., z=0), we can state that:

$$\mathbf{H}_2(\bar{r}_b) = \mathbf{H}_2(z=0) = H_{2x}(z=0)\,\hat{\mathbf{a}}_x + H_{2y}(z=0)\,\hat{\mathbf{a}}_y + H_{2z}(z=0)\,\hat{\mathbf{a}}_z$$

and:

$$\mathbf{B}_2(\bar{r}_b) = \mathbf{B}_2(z=0) = B_{2x}(z=0)\,\hat{\mathbf{a}}_x + B_{2y}(z=0)\,\hat{\mathbf{a}}_y + B_{2z}(z=0)\,\hat{\mathbf{a}}_z$$

Therefore, we need to find the **scalar components** $H_{2x}(z=0)$, $B_{2x}(z=0)$, etc.

First, we note that $\hat{\mathbf{a}}_z$ is **normal** to the interface, while $\hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$ are **tangential**.

Thus, from **boundary condition**:

$$\hat{a}_n \times \left(\mathbf{H}_1(\bar{r}_b) - \mathbf{H}_2(\bar{r}_b) \right) = \mathbf{J}_s(\bar{r}_b)$$

where we note that $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z$, we find:

$$\hat{\mathbf{a}}_z \times \left(\mathbf{H}_1(z=0) - \mathbf{H}_2(z=0) \right) = \mathbf{J}_s(z=0)$$

$$\hat{\mathbf{a}}_z \times \left[(3 - H_{2x})\hat{\mathbf{a}}_x + \left(0 - H_{2y}\right)\hat{\mathbf{a}}_y + (5 - H_{2z})\hat{\mathbf{a}}_z \right] = 2\,\hat{\mathbf{a}}_y$$

$$(3 - H_{2x})\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_x + \left(0 - H_{2y}\right)\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_y + (5 - H_{2z})\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 2\,\hat{\mathbf{a}}_y$$

$$(3 - H_{2x})\hat{\mathbf{a}}_y - \left(0 - H_{2y}\right)\hat{\mathbf{a}}_x = 2\,\hat{\mathbf{a}}_y$$

$$(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x = 2\,\hat{\mathbf{a}}_y$$

Thus, we can ascertain:

$$(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x = 2\,\hat{\mathbf{a}}_y$$

$$(3 - H_{2x})\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x + H_{2y}\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_x = 2\,\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x$$

$$H_{2y} = 0$$

and likewise:

$$(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x = 2\,\hat{\mathbf{a}}_y$$

$$(3 - H_{2x})\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_y = 2\,\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_y$$

$$3 - H_{2x} = 2$$

$$H_{2x} = 1$$

Therefore:

$$H_{2x}(z=0)=1 \qquad \text{and} \qquad H_{2y}(z=0)=0$$

Q: *But what about scalar component* $H_{2z}(z=0)$*?*

A: We can find it using our **second** boundary condition:

$$\mu_1 \mathbf{H}_{1n}(\bar{r}_b)=\mu_2 \mathbf{H}_{2n}(\bar{r}_b)$$

From which we find:

$$\begin{aligned}\mu_1 H_{1z}(z=0)\hat{\mathbf{a}}_z &= \mu_2 H_{2z}(z=0)\hat{\mathbf{a}}_z \\ 2\mu_0\, 5\,\hat{\mathbf{a}}_z &= 3\mu_0\, H_{2z}(z=0)\,\hat{\mathbf{a}}_z\end{aligned}$$

And therefore:

$$\begin{aligned}2\mu_0\, 5\,\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z &= 3\mu_0\, H_{2z}(z=0)\,\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z \\ 2\mu_0\, 5 &= 3\mu_0\, H_{2z}(z=0) \\ H_{2z}(z=0) &= \frac{10}{3}\end{aligned}$$

Thus, we find that:

$$\begin{aligned}\mathbf{H}_2(z=0) &= H_{2x}(z=0)\,\hat{\mathbf{a}}_x + H_{2y}(z=0)\,\hat{\mathbf{a}}_y + H_{2z}(z=0)\,\hat{\mathbf{a}}_z \\ &= \hat{\mathbf{a}}_x + {}^{10}\!/_{3}\,\hat{\mathbf{a}}_z\end{aligned}$$

And since:

$$\mathbf{B}_2(z=0) = \mu_2\,\mathbf{H}_2(z=0)$$

We find:

$$\begin{aligned}\mathbf{B}_2(z=0) &= 3\mu_0\left(\hat{\mathbf{a}}_x + 10/3\,\hat{\mathbf{a}}_z\right)\\ &= 3\mu_0\,\hat{\mathbf{a}}_x + 10\mu_0\,\hat{\mathbf{a}}_z\end{aligned}$$

Q: *But these are the values of the fields at the interface—what are the fields **throughout** region 2?*

A: Note that there are **no conduction currents** within region 2. Thus, we find **within region 2**:

$$\nabla \times \mathbf{H}_2(\bar{r}) = 0 \qquad (z<0)$$

Note that a **constant** magnetic field will satisfy the above equation. Moreover, the following **constant** magnetic field will **likewise** satisfy our **boundary condition** $\mathbf{H}_2(z=0)$:

$$\mathbf{H}_2(\bar{r}) = \hat{\mathbf{a}}_x + {}^{10}\!/_{3}\,\hat{\mathbf{a}}_z \qquad \left[{}^{A}\!/_{m}\right]$$

In other words, the value of the magnetic field at the boundary is **likewise** the value of the magnetic field **everywhere throughout** region 2 ($\mathbf{H}_2(\bar{r})$ is a **constant** vector field!). The **magnetic flux density** is therefore:

$$\mathbf{B}_2(\bar{r}) = \mu_2\,\mathbf{H}_2(\bar{r}) = 3\mu_0\,\hat{\mathbf{a}}_x + 10\mu_0\,\hat{\mathbf{a}}_z \qquad \left[W/m^2\right]$$